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► **To cite this version:**

Joël Blot, Constantin Buge, Philippe Cieutat. Local attractivity in nonautonomous semilinear evolution equations. *Nonautonomous dynamical systems*, 2014, 1 (1), 10.2478/msds-2014-0002 . hal-02173370

HAL Id: hal-02173370

<https://uvsq.hal.science/hal-02173370v1>

Submitted on 6 Jan 2025

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Joël Blot*, Constantin Bușe, and Philippe Cieutat

Local attractivity in nonautonomous semilinear evolution equations

Abstract: We study the local attractivity of mild solutions of equations in the form $u'(t) = A(t)u(t) + f(t, u(t))$, where $A(t)$ are (possible) unbounded linear operators in a Banach space and where f is a (possible) nonlinear mapping. Under conditions of exponential stability of the linear part, we establish the local attractivity of various kinds of mild solutions. To obtain these results we provide several results on the Nemytskii operators on the space of the functions which converge to zero at infinity.

Keywords: semilinear evolution equation, evolution family, exponential stability, attractivity, Nemytskii operator

MSC: 35B35, 35B40, 35B41, 35K58, 47D06

DOI 10.2478/msds-2014-0002

Received January 17, 2014; accepted March 21, 2014.

1 Introduction

From a family of linear unbounded operators $(A(t))_{t \in \mathbb{R}_+}$ on a Banach space X , from a function $b : \mathbb{R} \rightarrow X$ and from a nonlinear mappings $f : \mathbb{R}_+ \times X \rightarrow X$ we consider the following evolution equations:

$$u'(t) = A(t)u(t) \tag{1.1}$$

$$u'(t) = A(t)u(t) + b(t) \tag{1.2}$$

$$u'(t) = A(t)u(t) + f(t, u(t)). \tag{1.3}$$

We study the questions of the local attractivity and of the convergence to zero at infinity of mild solutions of these equations when their linear part is exponentially stable.

Now we describe the contents of the paper. In Section 2 we give the notation which is used in the paper. In Section 3 we study the properties of the Nemytskii operators on the spaces $C_0(\mathbb{R}_+, X)$, the spaces of the continuous functions on \mathbb{R}_+ which converge toward zero at infinity. We provide a necessary and sufficient condition for the continuity of such operators, we provide sufficient conditions for their Fréchet-differentiability, and we also give conditions to ensure that they are Lipschitzian. In the setting of the discrete time, the Nemytskii operators on the spaces $c_0(\mathbb{N}, X)$ are studied in [4]. Properties of the Nemytskii operators on other function spaces are given in [3].

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In Section 4 we establish conditions to ensure that all the mild solutions converge to zero at infinity.

In Section 5 we provide general conditions to ensure the local attractivity of a general mild solution of (1.3), after that using conditions on the partial differential of f along a given mild solution, we establish a theorem to ensure the local attractivity when this given mild solution is almost periodic, asymptotically almost periodic, or almost automorphic. In the last result of this section, using the implicit function theorem, we provide condition to ensure the local attractivity of a given mild solution which converge to zero at infinity.

2 Notation

When X and Y are Banach spaces, $C^0(X, Y)$ stands for the space of the continuous functions from X into Y , and $C^0(\mathbb{R}_+ \times X, Y)$ stands for the space of the continuous functions from $\mathbb{R}_+ \times X$ into Y . $C_0(\mathbb{R}_+, X) := \{u \in C^0(\mathbb{R}_+, X) : \lim_{t \rightarrow +\infty} u(t) = 0\}$. Endowed with the norm $\|u\|_\infty := \sup_{t \in \mathbb{R}_+} \|u(t)\|$, $C_0(\mathbb{R}_+, X)$ is a Banach space, [11] (Chapter 7). $C^1(X, Y)$ is the space of the continuously Fréchet-differentiable functions from X into Y , and $C^1(\mathbb{R}_+ \times X, Y)$ is the space of the continuously Fréchet-differentiable functions from $\mathbb{R}_+ \times X$ into Y .

$\mathcal{L}(X, Y)$ stands for the space of the linear continuous functions from X into Y .

When A is a linear unbounded operator on X , $\mathcal{D}(A)$ is the domain of A . When $f \in C^0(\mathbb{R}_+ \times X, X)$, when $(U(t, s))_{t \geq s \geq 0}$ is a well-defined evolution family associated to a family of unbounded linear operators on a Banach space X , and when, for all $x \in X$, there exists a unique mild solution $u_x \in C^0(\mathbb{R}_+, X)$ of the Cauchy problem ((1.3), $u(0) = x$), u_x satisfies

$$u_x(t) = U(t, 0)x + \int_0^t U(t, s)f(s, u_x(s))ds \quad (2.1)$$

for all $t \in \mathbb{R}_+$. In such a setting, a mild solution \bar{u} of (1.3) is so-called *locally attractive* when there exists $r \in (0, +\infty)$ such that

$$\|x - \bar{u}(0)\| \leq r \Rightarrow \lim_{t \rightarrow +\infty} \|u_x(t) - \bar{u}(t)\| = 0.$$

A mild solution \bar{u} is called *globally attractive* when $\lim_{t \rightarrow +\infty} \|u_x(t) - \bar{u}(t)\| = 0$ for all $x \in X$.

$AP^0(\mathbb{R}_+, X)$ denotes the space of the almost periodic functions (in the sense of Bohr) from \mathbb{R}_+ into X , [1], [8]. $AAP(\mathbb{R}_+, X)$ denotes the space of the asymptotically almost periodic functions (in the sense of Fréchet) from \mathbb{R}_+ into X , [14]. $AA(\mathbb{R}_+, X)$ denotes the space of the almost automorphic functions (in the sense of Bochner) from \mathbb{R}_+ into X , [8], [9]. Endowed with the norm $\|\cdot\|_\infty$, these three spaces are Banach spaces.

A mapping $f : \mathbb{R}_+ \times X \rightarrow Y$ is called almost periodic in t uniformly in x when f is continuous and, for all compact subset K in X , for all $\epsilon > 0$, there exists $\ell = \ell(\epsilon, K) > 0$ such that, for all $r \in \mathbb{R}_+$, there exists $\tau \in [r, r + \ell]$ satisfying $\|f(t + r + \tau, x) - f(t, x)\| \leq \epsilon$ for all $(t, x) \in \mathbb{R}_+ \times K$, [13], [3]. The set of such mappings is denoted by $APU(\mathbb{R}_+ \times X, Y)$.

A mapping $f : \mathbb{R}_+ \times X \rightarrow Y$ is called asymptotically almost periodic in t uniformly in x when f is continuous and, for all compact subset K in X , for all $\epsilon > 0$, there exist $T = T(\epsilon, K) \geq 0$ and $\ell = \ell(\epsilon, K) > 0$ such that for all $r \in \mathbb{R}_+$, there exists $\tau \in [r, r + \ell]$ satisfying $\|f(t + \tau, x) - f(t, x)\| \leq \epsilon$ for all $(t, x) \in [T, +\infty) \times K$, [14], [3]. The set of such mappings is denoted by $AAPU(\mathbb{R}_+ \times X, Y)$.

A mapping $f : \mathbb{R}_+ \times X \rightarrow Y$ is called almost automorphic in t uniformly in x when $f(\cdot, x) \in AA(\mathbb{R}_+, Y)$ for all $x \in X$, and when, for all compact subset K in X , for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon, K) > 0$ such that, for all $x, z \in K$, $\|x - z\| \leq \delta$ implies $\|f(t, x) - f(t, z)\| \leq \epsilon$ for all $t \in \mathbb{R}_+$, [3]. The set of such mappings is denoted by $AAU(\mathbb{R}_+ \times X, Y)$.

$\bar{B}(R) := \{x \in X : \|x\| \leq R\}$ is the closed ball centered at zero with a radius equal to R .

3 Nemytskii operators

Let X and Y be two Banach spaces.

Lemma 3.1. *Let $\phi \in C^0(\mathbb{R}_+ \times X, Y)$. The two following conditions are equivalent.*

- (i) $\lim_{(t,x) \rightarrow (+\infty, 0)} \phi(t, x) = 0$.
- (ii) For all $u \in C_0(\mathbb{R}_+, X)$, $[t \mapsto \phi(t, u(t))] \in C_0(\mathbb{R}_+, Y)$.

Proof. First we prove the implication ((i) \Rightarrow (ii)). Let $u \in C_0(\mathbb{R}_+, X)$. Since u is continuous on \mathbb{R}_+ , we have that $[t \mapsto (t, u(t))]$ is continuous, and since ϕ is continuous, $[t \mapsto \phi(t, u(t))]$ is continuous on \mathbb{R}_+ as a composition of continuous functions.

Note that the assertion (i) means:

$$\forall \epsilon > 0, \exists R_\epsilon > 0, \exists \eta_\epsilon > 0, \forall t \in \mathbb{R}_+, \forall x \in X, (t \geq R_\epsilon, \|x\| \leq \eta_\epsilon) \Rightarrow \|\phi(t, x)\| \leq \epsilon.$$

Since $\lim_{t \rightarrow +\infty} u(t) = 0$, we know that:

$$\forall \alpha > 0, \exists \beta_\alpha > 0, \forall t \in \mathbb{R}_+, (t \geq \beta_\alpha \Rightarrow \|u(t)\| \leq \alpha).$$

We arbitrarily fix $\epsilon > 0$ and we set $S_\epsilon := \max\{R_\epsilon, \beta_{\eta_\epsilon}\} > 0$. When $t \in \mathbb{R}_+$ satisfies $t \geq S_\epsilon$ then $t \geq R_\epsilon$ and $\|u(t)\| \leq \eta_\epsilon$ that implies $\|\phi(t, u(t))\| \leq \epsilon$. And so we have proven that $[t \mapsto \phi(t, u(t))] \in C_0(\mathbb{R}_+, Y)$.

Conversely to prove the implication ((ii) \Rightarrow (i)) we proceed by contradiction, we assume that (ii) holds and that (i) does not hold. Note that the negation of (i) is the following assertion: $\exists \epsilon > 0, \forall R > 0, \forall \eta > 0, \exists \hat{t}(R, \eta) \in \mathbb{R}_+, \exists \hat{x}(R, \eta) \in X$ such that $\hat{t}(R, \eta) \geq R$, $\hat{x}(R, \eta) \leq \eta$ and $\|\phi(\hat{t}(R, \eta), \hat{x}(R, \eta))\| > \epsilon$.

By induction we build two sequences by defining $t_1 := \hat{t}(1, 1)$ and $x_1 := \hat{x}(1, 1)$, and when $n > 1$, $t_{n+1} := \hat{t}(\max\{t_n + 1, n + 1\}, \frac{1}{n+1})$ and $x_{n+1} := \hat{x}(\max\{t_n + 1, n + 1\}, \frac{1}{n+1})$. And so we have built a sequence $(t_n)_n$ in \mathbb{R}_+ and a sequence $(x_n)_n$ in X such that $t_{n+1} > t_n$, $t_n \geq n$, $\|x_n\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}_*$. Then we have $\|\phi(t_n, x_n)\| > \epsilon$ for all $n \in \mathbb{N}_*$. Using these sequences we build the function $\varpi : \mathbb{R}_+ \rightarrow X$ by setting, for all $n \in \mathbb{N}_*$ and for all $t \in [t_n, t_{n+1}]$, $\varpi(t) := \frac{1}{t_{n+1} - t_n}((t - t_n)x_{n+1} + (t_{n+1} - t)x_n)$. Note that ϖ is piecewise affine and continuous on \mathbb{R}_+ . Note that, for all $n \in \mathbb{N}_*$ and for all $t \in [t_n, t_{n+1}]$, we have $\varpi(t) \in [x_n, x_{n+1}]$ that implies $\|\varpi(t)\| \leq \max\{\|x_n\|, \|x_{n+1}\|\} \leq \frac{1}{n}$. When we consider an $\alpha > 0$, we take $n_\alpha \in \mathbb{N}_*$ such that $n_\alpha \geq \frac{1}{\alpha}$. When $t \geq t_{n_\alpha}$ then there exists $n \in \mathbb{N}_*$, $n \geq n_\alpha$ such that $t \in [t_n, t_{n+1}]$, and then we have $\|\varpi(t)\| \leq \frac{1}{n} \leq \alpha$. That prove: $\lim_{t \rightarrow +\infty} \varpi(t) = 0$. After (ii) we know that $[t \mapsto \phi(t, \varpi(t))] \in C_0(\mathbb{R}_+, Y)$ and then we have $\lim_{t \rightarrow +\infty} \phi(t, \varpi(t)) = 0$. Since $\lim_{n \rightarrow +\infty} t_n = +\infty$, we have $\lim_{n \rightarrow +\infty} \phi(t_n, x_n) = \lim_{n \rightarrow +\infty} \phi(t_n, \varpi(t_n)) = 0$, and since $\|\phi(t_n, x_n)\| > \epsilon > 0$ for all $n \in \mathbb{N}_*$, we obtain a contradiction. \square

Now we can introduce the following conditions.

- (A1) $\phi \in C^0(\mathbb{R}_+ \times X, Y)$.
- (A2) $\lim_{(t,x) \rightarrow (+\infty, 0)} \phi(t, x) = 0$.

Under these two conditions, after Lemma 3.1 we can define the Nemytskii operator $N_\phi : C_0(\mathbb{R}_+, X) \rightarrow C_0(\mathbb{R}_+, X)$ by setting, for all $u \in C_0(\mathbb{R}_+, X)$,

$$N_\phi(u) := [t \mapsto \phi(t, u(t))].$$

Remark 3.2. *When $\phi \in C^0(X, Y)$, i.e. ϕ does not depend on t , condition (A2) becomes $\phi(0) = 0$ since, using the continuity of ϕ , we have $\phi(0) = \lim_{x \rightarrow 0} \phi(x) = 0$.*

Theorem 3.3. *Let $\phi : \mathbb{R}_+ \times X \rightarrow Y$ be a mapping. Under the conditions (A1) and (A2), we have $N_\phi \in C^0(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, Y))$.*

Proof. Using Lemma 3.1, we know that N_ϕ is well defined. We have yet seen that (A2) means

$$\left. \begin{array}{l} \forall \epsilon > 0, \exists R_\epsilon > 0, \exists \eta_\epsilon > 0, \forall t \in \mathbb{R}_+, \forall x \in X, \\ (t \geq R_\epsilon, \|x\| \leq \eta_\epsilon) \Rightarrow \|\phi(t, x)\| \leq \epsilon. \end{array} \right\} \quad (3.1)$$

We fix $u \in C_0(\mathbb{R}_+, X)$ and $\epsilon > 0$. Since $\lim_{t \rightarrow +\infty} u(t) = 0$ we know that there exists $T_\epsilon > 0$ such that $(t \geq T_\epsilon \Rightarrow \|u(t)\| \leq \frac{1}{2}\eta_{\epsilon/2})$ where $\eta_{\epsilon/2}$ is provided by (3.1). Let $v \in C_0(\mathbb{R}_+, X)$ such that $\|u - v\|_\infty \leq \frac{1}{2}\eta_{\epsilon/2}$. Then, for all $t \geq T_\epsilon$ we have

$$\|v(t)\| \leq \|u(t)\| + \|u - v\|_\infty \leq 2 \cdot \frac{1}{2} \eta_{\epsilon/2} = \eta_{\epsilon/2}.$$

We set $S_\epsilon := \max\{R_{\epsilon/2}, T_\epsilon\} > 0$ where $R_{\epsilon/2}$ is provided by (3.1). And so we have $\|u(t)\| \leq \eta_{\epsilon/2}$ and $\|v(t)\| \leq \eta_{\epsilon/2}$ for all $t \geq S_\epsilon$, and then using (3.1) we obtain $\|\phi(t, u(t))\| \leq \frac{\epsilon}{2}$ and $\|\phi(t, v(t))\| \leq \frac{\epsilon}{2}$ that implies

$$\|\phi(t, u(t)) - \phi(t, v(t))\| \leq \|\phi(t, u(t))\| + \|\phi(t, v(t))\| \leq 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

And so we have proven

$$\left. \begin{array}{l} \exists S_\epsilon > 0, \forall v \in C_0(\mathbb{R}_+, X), \\ \|u - v\|_\infty \leq \frac{1}{2}\eta_{\epsilon/2} \Rightarrow (\forall t \geq S_\epsilon, \|\phi(t, u(t)) - \phi(t, v(t))\| \leq \epsilon). \end{array} \right\} \quad (3.2)$$

After Lemma 3.10 in [3], the Nemytskii operator $N_\phi^1 : C^0([0, S_\epsilon], X) \rightarrow C^0([0, S_\epsilon], Y)$, defined by $N_\phi^1(\varphi) := [t \mapsto \phi(t, \varphi(t))]$, is continuous since the restriction of ϕ is continuous on $[0, S_\epsilon] \times X$. Then the following assertion holds.

$$\left. \begin{array}{l} \exists \delta_\epsilon > 0, \forall \varphi \in C^0([0, S_\epsilon], X), \\ \sup_{t \in [0, S_\epsilon]} \|u(t) - \varphi(t)\| \leq \delta_\epsilon \Rightarrow (\forall t \in [0, S_\epsilon], \|\phi(t, u(t)) - \phi(t, \varphi(t))\| \leq \epsilon). \end{array} \right\} \quad (3.3)$$

Now we set $\theta_\epsilon := \max\{\frac{1}{2}\eta_{\epsilon/2}, \delta_\epsilon\} > 0$. If $v \in C_0(\mathbb{R}_+, X)$ satisfies $\|u - v\|_\infty \leq \theta_\epsilon$, then, using (3.2) and (3.3), we obtain

$$\forall t \in [0, S_\epsilon], \|\phi(t, u(t)) - \phi(t, v(t))\| \leq \epsilon$$

and

$$\forall t \in [S_\epsilon, +\infty), \|\phi(t, u(t)) - \phi(t, v(t))\| \leq \epsilon$$

and consequently we obtain: $\forall t \in \mathbb{R}_+, \|\phi(t, u(t)) - \phi(t, v(t))\| \leq \epsilon$. To summarize we have proven

$$\left. \begin{array}{l} \forall u \in C_0(\mathbb{R}_+, X), \forall \epsilon > 0, \exists \theta_\epsilon > 0, \forall v \in C_0(\mathbb{R}_+, X), \\ \|u - v\|_\infty \leq \theta_\epsilon \Rightarrow \|N_\phi(u) - N_\phi(v)\|_\infty \leq \epsilon. \end{array} \right\}$$

that is the continuity of N_ϕ . □

Now, to treat the differentiability of the Nemytskii operator, we introduce the following list of conditions.

(A3) For all $(t, x) \in \mathbb{R}_+ \times X$, the Fréchet partial differential $D_2\phi(t, x)$ exists and $D_2\phi \in C^0(\mathbb{R}_+ \times X, \mathcal{L}(X, Y))$.

(A4) $\lim_{t \rightarrow +\infty} \phi(t, 0) = 0$.

(A5) $\lim_{(t,x) \rightarrow (+\infty, 0)} D_2\phi(t, x) = 0$.

Lemma 3.4. *Let $\phi : \mathbb{R}_+ \times X \rightarrow Y$ be a mapping which satisfies the conditions (A1), (A3), (A4) and (A5). Then ϕ satisfies (A2).*

Proof. (A5) means:

$$\left. \begin{array}{l} \forall \epsilon > 0, \exists R_\epsilon^1 > 0, \exists \eta_\epsilon^1 > 0, \forall t \in \mathbb{R}_+, \forall x \in X, \\ (t \geq R_\epsilon^1, \|x\| \leq \eta_\epsilon^1) \Rightarrow \|D_2\phi(t, x)\| \leq \epsilon. \end{array} \right\} \quad (3.4)$$

(A4) means:

$$\forall \epsilon > 0, \exists \xi_\epsilon > 0, \forall t \in \mathbb{R}_+, t \geq \xi_\epsilon \Rightarrow \|\phi(t, 0)\| \leq \epsilon. \quad (3.5)$$

We fix $\epsilon > 0$ and we set $\zeta_\epsilon := \max\{R_{\sqrt{\epsilon/2}}^1, \xi_{\epsilon/2}\} > 0$ and $\mu_\epsilon := \min\{\sqrt{\frac{\epsilon}{2}}, \eta_{\sqrt{\epsilon/2}}^1\} > 0$. Let $t \in \mathbb{R}_+$ such that $t \geq \zeta_\epsilon$ and let $x \in X$ such that $\|x\| \leq \mu_\epsilon$. Note that for all $z \in [0, x]$, we have $\|z\| \leq \|x\| \leq \mu_\epsilon$. Using the mean value inequality ([6]) we obtain

$$\|\phi(t, x)\| \leq \|\phi(t, x) - \phi(t, 0)\| + \|\phi(t, 0)\| \leq \sup_{z \in [0, x]} \|D_2\phi(t, z)\| \cdot \|x\| + \|\phi(t, 0)\| \leq \sqrt{\frac{\epsilon}{2}} \sqrt{\frac{\epsilon}{2}} + \frac{\epsilon}{2}.$$

And so we have proven the following assertion

$$\left. \begin{array}{l} \forall \epsilon > 0, \exists \zeta_\epsilon > 0, \exists \mu_\epsilon > 0, \forall t \in \mathbb{R}_+, \forall x \in X, \\ (t \geq \zeta_\epsilon, \|x\| \leq \mu_\epsilon) \Rightarrow \|\phi(t, x)\| \leq \epsilon \end{array} \right\}$$

that is (A2). □

Theorem 3.5. *Let $\phi : \mathbb{R}_+ \times X \rightarrow Y$. Under the conditions (A1), (A3), (A4) and (A5), $N_\phi \in C^1(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, Y))$ and for all $u, h \in C_0(\mathbb{R}_+, X)$, $DN_\phi(u).h = [t \mapsto D_2\phi(t, u(t)).h(t)]$.*

Proof. Using Lemma 3.4 and Theorem 3.3, we have $N_\phi \in C^0(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, Y))$, and using Theorem 3.3 on $D_2\phi$, we obtain that $N_{D_2\phi}$ is continuous from $C_0(\mathbb{R}_+, X)$ into $C_0(\mathbb{R}_+, \mathcal{L}(X, Y))$.

We fix $u \in C_0(\mathbb{R}_+, X)$. Then we have

$$\forall \epsilon > 0, \exists T_\epsilon > 0, \forall t \in \mathbb{R}_+, t \geq T_\epsilon \Rightarrow \|u(t)\| \leq \epsilon. \quad (3.6)$$

The meaning of (A5) is

$$\left. \begin{array}{l} \forall \epsilon > 0, \exists R_\epsilon^1 > 0, \exists \eta_\epsilon^1, \forall t \in \mathbb{R}_+, \forall x \in X, \\ (t \geq R_\epsilon^1, \|x\| \leq \eta_\epsilon^1) \Rightarrow \|D_2\phi(t, x)\| \leq \epsilon. \end{array} \right\} \quad (3.7)$$

For all $L \in (0, +\infty)$, since $\{(t, u(t)) : t \in [0, L]\}$ is compact, we can use the lemma of Heine-Schwartz ([12], Footnote (**), p. 355), and we can assert that the following assertion holds.

$$\left. \begin{array}{l} \forall L \in (0, +\infty), \forall \epsilon > 0, \exists \beta(L, \epsilon) > 0, \forall t \in [0, L], \forall x \in X, \\ \|x - u(t)\| \leq \beta(L, \epsilon) \Rightarrow \|D_2\phi(t, x) - D_2\phi(t, u(t))\| \leq \epsilon. \end{array} \right\} \quad (3.8)$$

We fix $\epsilon > 0$ and we set

$$\left. \begin{array}{l} S_\epsilon := \max\{R_{\epsilon/2}^1, T_{\eta_{\epsilon/2}}\} \\ y_\epsilon := \min\{\frac{1}{2}\eta_{\epsilon/2}^1, \beta(S_\epsilon, \epsilon)\}. \end{array} \right\} \quad (3.9)$$

Let $h \in C_0(\mathbb{R}_+, X)$ such that $\|h\|_\infty \leq y_\epsilon$. Then

$$t \geq S_\epsilon \Rightarrow (t \geq R_{\epsilon/2}^1, \|u(t)\| \leq \eta_{\epsilon/2}^1) \Rightarrow \|D_2\phi(t, u(t))\| \leq \frac{\epsilon}{2}.$$

Moreover $\|h\|_\infty \leq y_\epsilon \Rightarrow (\forall t \in \mathbb{R}_+, \|h(t)\| \leq \frac{1}{2}\eta_{\epsilon/2}^1)$.

When $t \geq S_\epsilon$ and $z \in [u(t), u(t) + h(t)]$ then we have $\|z\| \leq \max\{\|u(t)\|, \|u(t) + h(t)\|\} \leq \|u(t)\| + \|h(t)\| \leq \frac{1}{2}\eta_{\epsilon/2}^1 + \frac{1}{2}\eta_{\epsilon/2}^1 = \eta_{\epsilon/2}^1$ that implies

$$\|D_2\phi(t, z)\| \leq \frac{\epsilon}{2} \Rightarrow \|D_2\phi(t, z) - D_2\phi(t, u(t))\| \leq \|D_2\phi(t, z)\| + \|D_2\phi(t, u(t))\| \leq 2\frac{\epsilon}{2} = \epsilon.$$

And so we have proven the following assertion.

$$(t \geq S_\epsilon, z \in [u(t), u(t) + h(t)]) \Rightarrow \|D_2\phi(t, z) - D_2\phi(t, u(t))\| \leq \epsilon. \quad (3.10)$$

We take $L = S_\epsilon$ and from (3.8) we obtain:

$$(t \in [0, S_\epsilon], z \in [u(t), u(t) + h(t)]) \Rightarrow$$

$$\|z - u(t)\| \leq \|h(t)\| \leq y_\epsilon \leq \beta(S_\epsilon, \epsilon) \Rightarrow \|D_2\phi(t, z) - D_2\phi(t, u(t))\| \leq \epsilon.$$

And so we have proven

$$(t \in [0, S_\epsilon], z \in [u(t), u(t) + h(t)]) \Rightarrow \|D_2\phi(t, z) - D_2\phi(t, u(t))\| \leq \epsilon. \quad (3.11)$$

From (3.10) and (3.11) we deduce the following assertion.

$$\forall t \in \mathbb{R}_+, \sup_{z \in [u(t), u(t)+h(t)]} \|D_2\phi(t, z) - D_2\phi(t, u(t))\| \leq \epsilon. \quad (3.12)$$

We introduce $\Lambda : C_0(\mathbb{R}_+, X) \rightarrow C_0(\mathbb{R}_+, Y)$ the operator defined by $\Lambda(h) := [t \mapsto D_2\phi(t, u(t)).h(t)]$.

This operator is linear and it is continuous since $\sup_{t \in \mathbb{R}_+} \|D_2\phi(t, u(t))\| < +\infty$.

Using the mean value theorem ([6], Corollary 4.4, p. 342) and (3.12), we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} & \|\phi(t, u(t) + h(t)) - \phi(t, u(t)) - D_2\phi(t, u(t)).h(t)\| \\ & \leq \sup_{z \in [u(t), u(t)+h(t)]} \|D_2\phi(t, z) - D_2\phi(t, u(t))\| \cdot \|h(t)\| \leq \epsilon \|h(t)\|, \end{aligned}$$

that implies $\|N_\phi(u + h) - N_\phi(u) - \Lambda(h)\|_\infty \leq \epsilon \|h\|_\infty$. This last relation proves that N_ϕ is Fréchet differentiable at u and that $DN_\phi(u) = \Lambda$.

Now we consider $u, v, h \in C_0(\mathbb{R}_+, X)$ with $\|h\|_\infty \leq 1$. We have

$$\begin{aligned} \|DN_\phi(u).h - DN_\phi(v).h\|_\infty &= \sup_{t \in \mathbb{R}_+} \|D_2\phi(t, u(t)).h(t) - D_2\phi(t, v(t)).h(t)\| \\ &\leq \sup_{t \in \mathbb{R}_+} \|D_2\phi(t, u(t)) - D_2\phi(t, v(t))\| \cdot \|h(t)\| \leq \|N_{D_2\phi}(u) - N_{D_2\phi}(v)\| \end{aligned}$$

that implies

$$\|DN_\phi(u) - DN_\phi(v)\| \leq \|N_{D_2\phi}(u) - N_{D_2\phi}(v)\|,$$

and since $N_{D_2\phi}$ is continuous, DN_ϕ is also continuous. \square

After the continuity and the differentiability, we study the Lipschitzian property. We introduce the condition

$$(A6) \exists c \in \mathbb{R}_+, \forall t \in \mathbb{R}_+, \forall x, x_1 \in X, \|\phi(t, x) - \phi(t, x_1)\| \leq c \|x - x_1\|.$$

Theorem 3.6. *Let $\phi : \mathbb{R}_+ \times X \rightarrow Y$. Under (A1), (A4) and (A6), the Nemytskii operator N_ϕ is Lipschitzian from $C_0(\mathbb{R}_+, X)$ into $C_0(\mathbb{R}_+, Y)$.*

Proof. After Lemma 3.1, to prove that $N_\phi(C_0(\mathbb{R}_+, X)) \subset C_0(\mathbb{R}_+, Y)$, we need to prove that (A4) and (A6) imply (A2). This is easy when $c = 0$. Now we assume that $c > 0$. We fix $\epsilon > 0$. We use ξ_ϵ provided by (3.4). When $t \geq \xi_{\epsilon/2}$ and when $\|x\| \leq \frac{\epsilon}{2c}$ we obtain

$$\|\phi(t, x)\| \leq \|\phi(t, x) - \phi(t, 0)\| + \|\phi(t, 0)\| \leq c \|x\| + \|\phi(t, 0)\| \leq 2 \frac{\epsilon}{2} = \epsilon.$$

And so (A2) is fulfilled.

For all $u, v \in C_0(\mathbb{R}_+, X)$ and for all $t \in \mathbb{R}_+$, we have $\|\phi(t, u(t)) - \phi(t, v(t))\| \leq c \|u(t) - v(t)\|$ that implies $\|N_\phi(u) - N_\phi(v)\|_\infty \leq c \|u - v\|_\infty$. \square

4 Results on global asymptotic stability

We consider a family $(A(t))_{t \in \mathbb{R}_+}$ of linear unbounded operators on the Banach space X . We assume that this family generates an evolution family $(U(t, s))_{t \geq s \geq 0}$ on which we consider the following conditions.

(A7) $U(t, s) \in \mathcal{L}(X, X)$ for all $t \geq s \geq 0$.

(A8) $U(t, t) = I$ (the identity) for all $t \in \mathbb{R}_+$.

(A9) $U(t, r)U(r, s) = U(t, s)$ for all $t \geq r \geq s \geq 0$.

(A10) $(t, s) \mapsto U(t, s)x$ is continuous for all $x \in X$.

(A11) There exists $K \in \mathbb{R}_+$ and $\omega \in (0, +\infty)$ such that $\|U(t, s)\| \leq Ke^{-\omega(t-s)}$ for all $t \geq s \geq 0$.

Remark 4.1. After [10] (Theorem 8.1, p. 173) these conditions (A7-A11) can be seen as consequences of conditions on $(A(t))_{t \in \mathbb{R}_+}$. If $(A(t))_{t \in \mathbb{R}_+}$ satisfies the following conditions

(P1) $\mathcal{D}(A(t)) = D$ for all $t \in \mathbb{R}_+$, and D is dense in X .

(P2) For all $t \in \mathbb{R}_+$, the resolvent $R(\lambda : A(t))$ exists for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re} \lambda \leq 0$ and $:\exists M \in \mathbb{R}_+, \forall t \in \mathbb{R}_+, \forall \lambda \in \mathbb{C} \text{ s.t. } \operatorname{Re} \lambda \leq 0, \|R(\lambda : A(t))\| \leq \frac{M}{|\lambda|+1}$.

(P3) There exists $L \in \mathbb{R}_+$ and there exists $\alpha \in (0, 1]$ such that, for all $r, s, t \in \mathbb{R}_+, \|(A(t)-A(s))A(r)^{-1}\| \leq L|t-s|^\alpha$.

(P4) The operators $A(t)A(s)^{-1}$ are uniformly bounded for $s, t \in \mathbb{R}_+$ and there exists a closed operator $A(\infty)$ with domain D such that $\lim_{t \rightarrow +\infty} \|(A(t) - A(\infty))A(0)^{-1}\| = 0$.

Lemma 4.2. Under (A7-A11) the following assertions hold.

(i) 0 is globally asymptotically stable for (1.1).

(ii) For all $b \in C_0(\mathbb{R}_+, X)$ all the solutions of (1.2) converge to zero at infinity.

(iii) Following assertion (ii), when we fix $x \in X$, we can define the operator $\mathcal{S}_x : C_0(\mathbb{R}_+, X) \rightarrow C_0(\mathbb{R}_+, X)$, $\mathcal{S}_x(b) := [t \mapsto U(t, 0)x + \int_0^t U(t, s)b(s)ds]$. Then \mathcal{S}_x is affine continuous and it is $\frac{K}{\omega}$ -Lipschitzian.

Proof. We arbitrarily fix $x \in X$, $b \in C_0(\mathbb{R}_+, X)$ and $\epsilon > 0$. Since $b \in C_0(\mathbb{R}_+, X)$, there exists $T_\epsilon > 0$ such that for all $t \geq T_\epsilon$, $\|b(t)\| \leq \epsilon$. Then, for all $t \geq T_\epsilon$, we have

$$\begin{aligned} \|u(t)\| &\leq \|U(t, 0)x\| + \int_0^t \|U(t, s)\| \|b(s)\| ds \\ &\leq \|U(t, 0)x\| + K \cdot \int_0^t e^{-\omega(t-s)} \|b(s)\| ds \\ &\leq \|U(t, 0)x\| + K \cdot \int_0^{T_\epsilon} e^{-\omega t} e^{\omega s} \|b(s)\| ds + K \cdot \int_{T_\epsilon}^t e^{-\omega t} e^{\omega s} \epsilon ds \\ &= \|U(t, 0)x\| + K \cdot e^{-\omega t} \int_0^{T_\epsilon} e^{\omega s} \|b(s)\| ds + \epsilon \cdot K \cdot \frac{1}{\omega} (1 - e^{-\omega t} e^{\omega T_\epsilon}) \end{aligned}$$

and so when $t \rightarrow +\infty$, since $K \cdot \int_0^{T_\epsilon} e^{\omega s} \|b(s)\| ds$ is constant with respect to t , we have

$$\lim_{t \rightarrow +\infty} (K \cdot e^{-\omega t} \int_0^{T_\epsilon} e^{\omega s} \|b(s)\| ds) = 0,$$

and since $\lim_{t \rightarrow +\infty} \epsilon \frac{K}{\omega} (1 - e^{-\omega t} e^{\omega T_\epsilon}) = \epsilon \frac{K}{\omega}$, we obtain

$$\limsup_{t \rightarrow +\infty} \|u(t)\| \leq 0 + \epsilon \frac{K}{\omega},$$

and then taking $\epsilon \rightarrow 0+$, we obtain $\lim_{t \rightarrow +\infty} u(t) = 0$. And so the assertion (ii) is proven.

Taking $b = 0$ in assertion (ii), we obtain that 0 is an attractor for equation (1.1).

About assertion (iii), it is clear that \mathcal{S}_x is affine. We consider $b, b_1 \in C_0(\mathbb{R}_+, X)$, and then, for all $t \in \mathbb{R}_+$, we have

$$\begin{aligned} \|\mathcal{S}_x(b)(t) - \mathcal{S}_x(b_1)(t)\| &= \left\| \int_0^t U(t, s)(b(s) - b_1(s)) ds \right\| \leq \int_0^t Ke^{-\omega t} e^{\omega s} ds \cdot \|b - b_1\|_\infty \\ &= Ke^{-\omega t} \left(\frac{e^{\omega t}}{\omega} - \frac{1}{\omega} \right) \|b - b_1\|_\infty = \frac{K}{\omega} (1 - e^{-\omega t}) \|b - b_1\|_\infty \leq \frac{K}{\omega} \|b - b_1\|_\infty. \end{aligned}$$

□

Theorem 4.3. *Let $f : \mathbb{R}_+ \times X \rightarrow X$ a mapping which satisfies (A1), (A4) and (A6). Under (A7-A11), we assume that $c < \frac{\omega}{K}$ where c is provided by (A6) and K and ω are provided by (A11). Then all the mild solutions of equation (1.3) converge to zero at infinity.*

Proof. We arbitrarily fix a mild solution u of (1.3), we set $x := u(0)$. Note that $v \in C_0(\mathbb{R}_+, X)$ is a mild solution of the Cauchy problem ((1.3), $v(0) = x$) if and only if $v = S_x \circ N_f(v)$.

Using Theorem 3.6, we know that N_f is c -Lipschitzian from $C_0(\mathbb{R}_+, X)$ into $C_0(\mathbb{R}_+, X)$, and using Lemma 4.2, we know that S_x is $\frac{K}{\omega}$ -Lipschitzian from $C_0(\mathbb{R}_+, X)$ into $C_0(\mathbb{R}_+, X)$. And so $S_x \circ N_f$ is $c\frac{K}{\omega}$ -Lipschitzian from $C_0(\mathbb{R}_+, X)$ into $C_0(\mathbb{R}_+, X)$, and since $c < \frac{\omega}{K}$, $S_x \circ N_f$ is a strict contraction. Then we can use the Picard-Banach fixed point theorem that there exists a unique fixed point of $S_x \circ N_f$ that we denote by v . Using the uniqueness of the mild solution of a Cauchy problem, we can say that $u = v$, and so $u \in C_0(\mathbb{R}_+, X)$. \square

5 Results on local attractivity

Theorem 5.1. *Under (A7-A11) we assume that the mapping $f : \mathbb{R}_+ \times X \rightarrow X$ satisfies (A1). Let $\bar{u} \in C^0(\mathbb{R}_+, X)$ be a mild solution of (1.3) which satisfies the following conditions*

- (i) *There exist $R \in (0, +\infty)$ and $\sigma \in (0, 1]$ such that, for all $t \in \mathbb{R}_+$ and for all $y, y_1 \in \bar{B}(R)$, $\|f(t, \bar{u}(t) + y) - f(t, \bar{u}(t) + y_1)\| \leq \sigma \|y - y_1\|$.*
- (ii) *$\sigma < \frac{\omega}{K}$.*

Then \bar{u} is locally attractive for (1.3).

Proof. We introduce the mapping $h : \mathbb{R}_+ \times X \rightarrow X$ by setting

$$h(t, y) := f(t, \bar{u}(t) + y) - f(t, \bar{u}(t)). \quad (5.1)$$

Under assumptions (i) and (ii) we have

$$\forall (t, y) \in \mathbb{R}_+ \times \bar{B}(R), \|h(t, y)\| \leq \sigma \|y\| \leq \|y\| \leq R. \quad (5.2)$$

Since $h(t, y) - h(t, y_1) = f(t, \bar{u}(t) + y) - f(t, \bar{u}(t) + y_1)$, from assumption (i) we obtain

$$\forall t \in \mathbb{R}_+, \forall y, y_1 \in \bar{B}(R), \|h(t, y) - h(t, y_1)\| \leq \sigma \|y - y_1\|. \quad (5.3)$$

Using (5.1), (5.2), ((5.3), we can define $N_h : C_0(\mathbb{R}_+, \bar{B}(R)) \rightarrow C_0(\mathbb{R}_+, \bar{B}(R))$ by setting $N_h(v) := [t \mapsto h(t, v(t))]$. From (5.3) we obtain

$$\forall v, v_1 \in C_0(\mathbb{R}_+, \bar{B}(R)), \|N_h(v) - N_h(v_1)\|_\infty \leq \sigma \|v - v_1\|_\infty. \quad (5.4)$$

Using Lemma 4.2 we know that S_x is $\frac{K}{\omega}$ -Lipschitzian. When $y \in \bar{B}(R)$ and when $v \in C_0(\mathbb{R}_+, \bar{B}(R))$, noting that $h(t, 0) = 0$ and using Lemma 4.2, assumption (ii) and (5.2) we obtain

$$\|S_y(N_h(v))\|_\infty \leq \frac{K}{\omega} \|N_h(v)\|_\infty \leq \frac{K}{\omega} \sigma \|v\|_\infty \leq \|v\|_\infty \leq R,$$

that implies

$$S_y \circ N_h(C_0(\mathbb{R}_+, \bar{B}(R))) \subset C_0(\mathbb{R}_+, \bar{B}(R)). \quad (5.5)$$

Since the uniform convergence implies the pointwise convergence, $C_0(\mathbb{R}_+, \bar{B}(R))$ is a closed subset of the Banach space $C_0(\mathbb{R}_+, X)$, and then endowed with the distance $d_\infty(v, v_1) := \|v - v_1\|_\infty$, it is a complete metric space. Using Lemma 4.2, (5.4) and assumption (ii), $S_y \circ N_h$ is a strict contraction on $C_0(\mathbb{R}_+, \bar{B}(R))$. Then using the Picard-Banach fixed point theorem, we can assert that, for all $y \in \bar{B}(R)$, there exists a unique

$v_y \in C_0(\mathbb{R}_+, \overline{B}(R))$ such that $S_y \circ N_h(v_y) = v_y$. Using the uniqueness of the mild solution of the Cauchy problem on

$$v'(t) = A(t)v(t) + h(t, v(t)), \quad (5.6)$$

we obtain that, for all $y \in \overline{B}(R)$, the unique mild solution of the Cauchy problem ((5.6), $v(0) = y$) belongs to $C_0(\mathbb{R}_+, \overline{B}(R)) \subset C_0(\mathbb{R}_+, X)$.

We set $\bar{x} := \bar{u}(0)$. Let $x \in X$ such that $\|x - \bar{x}\| \leq R$. Then $x = \bar{x} + y$ with $y \in \overline{B}(R)$, and we denote by $u_{\bar{x}+y}$ the unique mild solution of ((1.3), $u(0) = \bar{x} + y$). We set $v := u_{\bar{x}+y} - \bar{u}$, then we have, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} v(t) &= u_{\bar{x}+y}(t) - \bar{u}(t) = \\ &= U(t, 0)(\bar{x} + y) + \int_0^t U(t, s)f(s, \bar{u}(s) + v(s))ds - U(t, 0)\bar{x} - \int_0^t U(t, s)f(s, \bar{u}(s))ds \\ &= U(t, 0)y + \int_0^t U(t, s)h(s, v(s))ds, \end{aligned}$$

and so v is the unique mild solution of ((5.6), $v(0) = y$) that implies that $v \in C_0(\mathbb{R}_+, \overline{B}(R))$, and consequently $\lim_{t \rightarrow +\infty} \|v(t)\| = 0$, i.e. $\lim_{t \rightarrow +\infty} \|u_{\bar{x}+y}(t) - \bar{u}(t)\| = 0$. \square

Remark 5.2. *If we assume that $D_2f(t, x)$ exists for all $(t, x) \in \mathbb{R}_+ \times X$, and when the following inequality holds*

$$\sup_{y \in \overline{B}(R)} \sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t) + y)\| \leq \sigma, \quad (5.7)$$

the assumption (i) of Theorem 5.1 is fulfilled. This is a simple consequence of the mean value theorem ([6, Corollary 4.3, p. 342], since for all $y, y_1 \in \overline{B}(R)$ and for all $t \in \mathbb{R}_+$,

$$\|f(t, \bar{u}(t) + y) - f(t, \bar{u}(t) + y_1)\| \leq \sup_{\theta \in [0, 1]} \|D_2f(t, \bar{u}(t) + (1 - \theta)y + \theta y_1)\| \cdot \|y - y_1\|,$$

and since $\overline{B}(R)$ is convex, we obtain

$$\|f(t, \bar{u}(t) + y) - f(t, \bar{u}(t) + y_1)\| \leq \sup_{z \in \overline{B}(R)} \|D_2f(t, \bar{u}(t) + z)\| \cdot \|y - y_1\|,$$

and so (5.7) implies (i) of Theorem 5.1.

Corollary 5.3. *We assume that (A7-A11) are fulfilled. We also assume that $f \in APU(\mathbb{R}_+ \times X, X)$ (respectively $f \in AAPU(\mathbb{R}_+ \times X, X)$, respectively $f \in AAU(\mathbb{R}_+ \times X, X)$, that $D_2f(t, x)$ exists for all $(t, x) \in \mathbb{R}_+ \times X$, and that $D_2f \in APU(\mathbb{R}_+ \times X, \mathcal{L}(X, X))$ (respectively $D_2f \in AAPU(\mathbb{R}_+ \times X, \mathcal{L}(X, X))$, respectively $D_2f \in AAU(\mathbb{R}_+ \times X, \mathcal{L}(X, X))$). Let $\bar{u} \in AP^0(\mathbb{R}_+, X)$ (respectively $\bar{u} \in AAP^0(\mathbb{R}_+, X)$, respectively $\bar{u} \in AA^0(\mathbb{R}_+, X)$) be a mild solution of (1.3) which satisfies the following inequality:*

$$\sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t))\| < \min\left\{\frac{\omega}{K}, 1\right\}.$$

Then \bar{u} is locally attractive for (1.3).

Proof. First we consider the case where the given solution is almost periodic. Using Theorem 3.5 in [3] on D_2f , the Nemytskii operator N_{D_2f} is continuous from $AP^0(\mathbb{R}_+, X)$ into $AP^0(\mathbb{R}_+, \mathcal{L}(X, X))$, therefore we have

$$\left. \begin{aligned} \forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall w \in AP^0(\mathbb{R}_+, X), \|w\|_\infty \leq \delta_\epsilon \Rightarrow \\ \forall t \in \mathbb{R}_+, \|D_2f(t, \bar{u}(t) + w(t)) - D_2f(t, \bar{u}(t))\| \leq \epsilon \end{aligned} \right\}$$

and then, for all $t \in \mathbb{R}_+$,

$$\|D_2f(t, \bar{u}(t) + w(t))\| \leq \|D_2f(t, \bar{u}(t) + w(t)) - D_2f(t, \bar{u}(t))\| + \|D_2f(t, \bar{u}(t))\| \leq \epsilon + \|D_2f(t, \bar{u}(t))\|.$$

Since the constant functions belong to $AP^0(\mathbb{R}_+, X)$, we can say that

$$\left. \begin{aligned} \forall \epsilon > 0, \exists \delta_\epsilon > 0, \forall y \in X, \|y\| \leq \delta_\epsilon \Rightarrow \\ \forall t \in \mathbb{R}_+, \|D_2f(t, \bar{u}(t) + y)\| \leq \epsilon + \|D_2f(t, \bar{u}(t))\|, \end{aligned} \right\}$$

i.e.

$$\left. \begin{aligned} \forall \epsilon > 0, \exists \delta_\epsilon > 0, \\ \sup_{y \in \bar{B}(\delta_\epsilon)} \sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t) + y)\| \leq \epsilon + \sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t))\|, \end{aligned} \right\} \quad (5.8)$$

We choose $\epsilon_0 := \frac{1}{2}(\min\{\frac{\omega}{K}, 1\} - \sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t))\|) > 0$, $R := \delta_{\epsilon_0}$, $\sigma := \epsilon_0 + \sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t))\| < \min\{\frac{\omega}{K}, 1\}$, and using (5.8) we see that (5.7) is fulfilled and using Remark 5.2 we obtain that the assumptions (i) and (ii) of Theorem 5.1 are fulfilled, and then using Theorem 5.1 we obtain the announced conclusion. To prove the case where the given solution is asymptotically almost periodic (respectively almost automorphic) it suffices to replace the use of Theorem 3.5 of [3] by the use of Theorem 8.4 (respectively Theorem 9.6 in [3]). \square

Remark 5.4. We can find in [2] conditions to ensure the existence of almost periodic mild solution of (1.3) when (A7-A11) are fulfilled and when $f \in APU(\mathbb{R}_+ \times X, X)$. About the existence of almost automorphic mild solution of (1.3) when $(A(t))_{t \in \mathbb{R}_+}$ is exponentially stable and when $f \in AAU(\mathbb{R}_+ \times X, X)$ we can see [5].

Remark 5.5. When $\bar{u} \in AP^0(\mathbb{R}_+, X)$ (respectively $AA^0(\mathbb{R}_+, X)$) the previous corollary says that the solutions which are near to \bar{u} are the sum of \bar{u} and of a function which belongs to $C_0(\mathbb{R}_+, X)$, and so they belong to $AAP^0(\mathbb{R}_+, X)$ (respectively $AAA^0(\mathbb{R}_+, X)$).

In Theorem 5.1 we have treated the case of a arbitrary mild solution, in Corollary 5.3 we have treated the cases of an almost periodic or an asymptotically almost periodic or an almost automorphic mild solution; in the following theorem we treat the case of a mild solution which converges to zero at infinity by using the implicit function theorem.

Theorem 5.6. We assume that (A7-A11) are fulfilled. Let $f : \mathbb{R}_+ \times X \rightarrow X$ be a mapping which satisfies the following conditions:

- (a) $f \in C^0(\mathbb{R}_+ \times X, X)$
- (b) $\forall (t, x) \in \mathbb{R}_+ \times X, D_2f(t, x)$ exists and $D_2f \in C^0(\mathbb{R}_+ \times X, \mathcal{L}(X, X))$
- (c) $\lim_{t \rightarrow +\infty} f(t, 0) = 0$
- (d) $\lim_{(t,x) \rightarrow (+\infty, 0)} D_2f(t, x) = 0$.

Let $\bar{u} \in C_0(\mathbb{R}, X)$ be a mild solution of (1.3). We set $\bar{x} := \bar{u}(0)$. We assume that $\sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t))\| < \frac{\omega}{K}$. Then \bar{u} is locally attractive for (1.3).

Proof. We introduce the nonlinear operator $\Gamma : X \times C_0(\mathbb{R}_+, X) \rightarrow C_0(\mathbb{R}_+, X)$ by setting

$$\Gamma(x, u) := u - S_x \circ N_f(u) \quad (5.9)$$

when $x \in X$ and $u \in C_0(\mathbb{R}_+, X)$. In a first step we establish that Γ is of class C^1 .

Since $S_x \circ N_f(u) = [t \mapsto U(t, 0)x + \int_0^t U(t, s)f(s, u(s))ds]$, we see that $x \mapsto \Gamma(x, u)$ is affine continuous and that

$$D_1\Gamma(x, u).y = [t \mapsto -U(t, 0).y] \quad (5.10)$$

for all $x, y \in X$ and for all $u \in C_0(\mathbb{R}_+, X)$. Since $D_1\Gamma(x, u)$ is independent of (x, u) , it is continuous and so we obtain

$$(x, u) \mapsto D_1\Gamma(x, u) \in C^0(X \times C_0(\mathbb{R}_+, X), \mathcal{L}(X, C_0(\mathbb{R}_+, X))). \quad (5.11)$$

Under Theorem 3.5 we can use our assumption to ensure that N_f is of class C^1 and $DN_f(u).h = [t \mapsto D_2f(t, u(t)).h(t)]$. Since S_x is affine continuous, it is of class C^1 , and so $S_x \circ N_f$ is C^1 as a composition of

C^1 operators, and since the identity is C^1 , $D_2\Gamma(x, u)$ exists and we have

$$D_2\Gamma(x, u) = I - \mathcal{S}_0 \circ DN_f(u) \quad (5.12)$$

for all $(x, u) \in X \times C_0(\mathbb{R}_+, X)$. Note that $D_2\Gamma(x, u)$ is independent of x , and since DN_f is of class C^1 , we obtain the following assertion.

$$(x, u) \mapsto D_2\Gamma(x, u) \in C^0(X \times C_0(\mathbb{R}_+, X), \mathcal{L}(C_0(\mathbb{R}_+, X), C_0(\mathbb{R}_+, X))). \quad (5.13)$$

Using Theorem 7.1 in [6] (p. 352), from (5.11) and (5.13) we deduce that Γ is of class C^1 . Since \mathcal{S}_0 is linear continuous, using Lemma 4.2, (iii), we know that $\|\mathcal{S}_0\| \leq \frac{K}{\omega}$, and $\|\mathcal{S}_0 \circ DN_f(\bar{u})\| \leq \|\mathcal{S}_0\| \cdot \sup_{t \in \mathbb{R}_+} \|D_2f(t, \bar{u}(t))\| < 1$ after our assumption, and so using Theorem 2.1 in [6] (p. 74) we can assert that $D_2\Gamma(\bar{x}, \bar{u}) = I - \mathcal{S}_0 \circ DN_f(\bar{u})$ is a topological isomorphism on $C_0(\mathbb{R}_+, X)$. Then we can use the implicit function theorem ([6], Theorem 2.1, p. 364) and we obtain that there exist an open neighborhood \mathcal{N} of \bar{x} in X , an open neighborhood \mathcal{V} of \bar{u} in $C_0(\mathbb{R}_+, X)$, and a C^1 mapping $\Phi : \mathcal{N} \rightarrow \mathcal{V}$ such that $\Phi(\bar{x}) = \bar{u}$ and such that $\Gamma(x, \Phi(x)) = 0$ for all $x \in \mathcal{N}$. Note that

$$\Gamma(x, \Phi(x)) = 0 \iff \Phi(x) = \mathcal{S}_x \circ N_f(\Phi(x)) \Rightarrow \Phi(x) = [t \mapsto U(t, 0)x + \int_0^t U(t, s)f(s, \Phi(x)(s))ds]$$

that implies that $\Phi(x)$ is a mild solution of the Cauchy problem ((1.3), $u(0) = x$). Using the uniqueness of the mild solutions of the Cauchy problems on (1.3), we can say that $u_x = \Phi(x)$ for all $x \in \mathcal{N}$. Consequently, for all $x \in \mathcal{N}$, $u_x \in C_0(\mathbb{R}_+, X)$, and then $\lim_{t \rightarrow +\infty} u_x(t) = 0$ and also $\lim_{t \rightarrow +\infty} (\bar{u}(t) - u_x(t)) = 0$, that is the local attractivity of \bar{u} . \square

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